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THE INVERSE STABLE RANGE FUNCTOR

ROBERT S. Y. WONG

ABSTRACT. We give an inverse construction of the stable range for general flows which may or may not admit an invariant measure. The inverse map is then shown to be a right inverse functor of the stable range functor.

The Krieger theorem asserts that in the type III_0 case, there is a one-to-one correspondence between orbit equivalent classes of ergodic transformations, algebraic isomorphism classes of Krieger factors, and conjugate isomorphism classes of nontransitive ergodic flows. This correspondence is established via explicit maps between the objects, namely the stable range map, the crossed product, and the flow of weights. In this paper we are concerned with inverting the stable range map, i.e., given a flow on a standard measure space, we will explicitly construct a transformation with the flow as its stable range. Since orbit equivalent transformations give rise to conjugate flows under the stable range map, the choice of the transformation is not unique. However, by using some kind of skew product with the odometer, it is possible to choose a natural one which satisfies the requirement. This construction provides the vital missing arrow which points in the opposite direction of the other maps.

For the case where the flow admits an invariant measure, such a transformation can be found in [4] or [5], but for a general flow the problem is more difficult. In [3] Hamachi showed that for a given ergodic flow, one can construct an ergodic action of $\mathbb{Z} \times (\mathbb{Z} + \alpha\mathbb{Z})$ on a certain standard measure space whose stable range is isomorphic to the flow. The result of Connes, Feldman, and Weiss guarantees that the equivalence relation generated by this action is generated by a single transformation T, but it is far from clear how to construct such a transformation explicitly. Noting that the stable range map is in fact a functor from the category of transformations with orbit transporting isomorphisms to the category of flows with conjugations, we see that the construction below also extends to the construction of a right inverse functor to the stable range functor. This fact establishes that the mod homomorphism from the automorphism group of a measured equivalence relation to the automorphism group of its associated flow is split (this also follows from Hamachi's construction [3, p. 399]). Our method also yields a transparent proof of surjectivity of mod, a fact noted by both Hamachi and Golodets [1].

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Preliminaries

A flow on an abelian von Neumann algebra A is a continuous action of $\mathbb{R}$ on A, i.e., a homomorphism $\alpha\colon \mathbb{R} \to \mathrm{Aut}(A)$, the automorphism group of A, such that, writing α_t for $\alpha(t)$, we have for any $x \in A$, the map $t \to \alpha_t(x)$ is σ-weakly continuous.

We write (α_t, A) for a flow on a von Neumann algebra. The flow is said to be ergodic if for $a \in A$, $\alpha_t(a) = a$ for all t implies that a is a multiple of the identity. We denote by $\mathscr{A}$ the category of ergodic flows on nonatomic abelian von Neumann algebras with separable preduals, with conjugations as morphisms.

Let A, B be nonatomic abelian von Neumann algebras with separable preduals. Let $\phi\colon A \to B$ be isomorphisms. Suppose there are standard measure spaces (X, μ), (Y, ν), an isomorphism $\hat{\phi}\colon (X, \mu) \to (Y, \nu)$, and isomorphisms $\sigma_1\colon A \to L^\infty(X, \mu)$, $\sigma_2\colon B \to L^\infty(Y, \nu)$ such that $(\sigma_2\phi)(a) = \sigma_1(a) \circ \hat{\phi}^{-1}$ for all $a \in A$. Then $\hat{\phi}$ is called a point realization of the map ϕ via σ_1 and σ_2.

Similarly if (α_t, A) belongs to $\mathscr{A}$, and there exists a flow (F_t, Ω, ν) and an isomorphism $\sigma\colon A \to L^\infty(\Omega, \nu)$ such that F_t is a point realization of α_t via σ for all t, then (F_t, Ω, ν) is called the point realization of α_t via σ. By a theorem of Mackey [8], given such a σ, the point realization exists and is unique.

Any two point realizations of (α_t, A) are conjugate isomorphic. If $\phi\colon A \to B$, $\psi\colon B \to C$ are morphisms in $\mathscr{A}$, and $\hat{\phi}$, $\hat{\psi}$ are point realizations of ϕ and ψ via some fixed isomorphisms, then $\hat{\phi} \circ \hat{\psi}$ is the point realization of $\phi \circ \psi$ via these isomorphisms. Thus point realization preserves commutative diagrams.

Let (T, X, μ) be a dynamical system. We can construct a flow, called the stable range of the transformation as follows. Let ds be the Lebesgue measure on $\mathbb{R}$. Define $\widetilde{T}$ on $(X \times \mathbb{R}, \mu \times e^{-s}\, ds)$ by

$$\widetilde{T}(x, r) = \left(Tx, r + \log \frac{d\mu \circ T}{d\mu}(x)\right).$$

$\widetilde{T}$ is in general nonergodic even though T may be ergodic. There is an action $\alpha_{\widetilde{T}}$ on $L^\infty(X \times \mathbb{R})$ induced from $\widetilde{T}$:

$$\alpha_{\widetilde{T}}(f)(x, r) = f(\widetilde{T}^{-1}(x, r)), \qquad f \in L^\infty(X \times \mathbb{R}).$$

We consider the fixed point subalgebra of $L^\infty(X \times \mathbb{R})$ under $\alpha_{\widetilde{T}}$, i.e., the algebra $L^\infty(X \times \mathbb{R})^{\widetilde{T}}$ of functions f such that $\alpha_{\widetilde{T}}(f) = f$. There exists an action F of $\mathbb{R}$ on $L^\infty(X \times \mathbb{R})$ given by

$$F_t(f)(x, r) = f(x, r + t), \qquad t \in \mathbb{R}.$$

This action commutes with $\alpha_{\widetilde{T}}$, i.e., $F_t \circ \alpha_{\widetilde{T}} = \alpha_{\widetilde{T}} \circ F_t$ for all t, and therefore leaves $L^\infty(X \times \mathbb{R})^{\widetilde{T}}$ invariant. The stable range of (T, X, μ) is any point realization of the restriction of F_t to $L^\infty(X \times \mathbb{R})^{\widetilde{T}}$. It is unique up to conjugations and is ergodic if and only if (T, X, μ) is ergodic.

The stable range can be thought of as a conjugacy class of flows, but we will choose a representative for each ergodic system as in the following.

The stable range map is in fact a functor from the category $\mathscr{X}$ of ergodic dynamical systems with orbit transporting transformations to the category $\mathscr{F}$ of ergodic flows with conjugations. First we define a functor S_1 from $\mathscr{A}$ to $\mathscr{F}$. Fix a standard measure space (I, m). For (α_t, A) in $\mathscr{A}$, choose an isomorphism $\sigma_A: A \to L^\infty(I, m)$, and define $S_1(\alpha_t, A)$ to be the point realization of (α_t, A) on (I, m) via σ.

Let $\phi: (\alpha_t, A) \to (\beta_t, B)$ be a conjugation. $S_1(\phi)$ is defined to be the automorphism on (I, m) which is the point realization of ϕ via σ_A, σ_B. It is clear that $S_1(\phi)$ conjugates between $S_1(\alpha_t, A)$ and $S_1(\beta_t, B)$ and that S_1 is a functor.

Let (T, X, μ) be an ergodic transformation, then by the construction of the stable range, we obtain a flow α_t on the abelian von Neumann algebra $L^\infty(X \times \mathbb{R})^{\widetilde{T}}$. This association is a functor S_2 from $\mathscr{X}$ to $\mathscr{A}$. For if $\phi: (T, X, \mu) \to (S, Y, \nu)$ is orbit transporting, then the map $\tilde{\phi}: (X \times \mathbb{R}) \to (Y \times \mathbb{R})$, given by

$$\tilde{\phi}(x, r) = \left(\phi(x), r + \log \frac{d\nu}{d\phi^*\mu}(\phi(x))\right)$$

is also orbit transporting from $\widetilde{T}$ to $\widetilde{S}$. Hence it restricts to the fixed point subalgebras. Clearly it conjugates between the flows so that S_2 is a functor. The stable range functor is defined to be $S_1 \circ S_2$.

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Let $\mathbb{Z}_5$ be the cyclic group of 5 elements, and let $\mathbf{X} = \prod_1^\infty \mathbf{Z}_5$. Let β_i, $i = 1, 2, 3, 4$, be positive real numbers such that $\log \beta_1$, $\log \beta_2$, $\log \beta_3$, and $\log \beta_4$ are rationally independent. Define a probability measure p on $\mathbf{X}$ by $p = \prod_1^\infty q$, where $q = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ is the probability measure on $\mathbf{Z}_5$ which gives mass α_i to point i, with ratios $\alpha_i/\alpha_{i+1} = \beta_i$, $i = 1, 2, 3, 4$. We let T be the usual odometer transformation on $\mathbf{X}$.

Lemma 1.1. *There exists integer valued Borel functions $n_i(x)$, $i = 1, 2, 3, 4$, such that*

$$\frac{dp \circ T}{dp}(x) = \beta_1^{n_1(x)} \beta_2^{n_2(x)} \beta_3^{n_3(x)} \beta_4^{n_4(x)}, \quad a.e.\ x \in \mathbf{X}.$$

This set of n_i is uniquely determined by the equation, and is the same regardless of the choice of β_i, subject to the rationally independent condition mentioned above.

Proof. Delete the point of null measure $(4, 4, \ldots)$ from $\mathbf{X}$. Then the collection $\mathscr{B}$ of sets of the form $((x_1, x_2, \ldots, x_n) \times \prod_{n+1}^\infty \mathbf{Z}_5$, with $n \geq 1$ and not all of the x_i, $i = 1, 2, \ldots, n$, equal to 4 generates the product Borel structure on $\mathbf{X}$. For $A \in \mathscr{B}$ and $x \in A$, define $n_i(x)$, $i = 1, 2, 3, 4$, by the formula

$$\frac{p(T(A))}{p(A)} = \prod_{i=1}^4 \beta_i^{n_i(x)}.$$

It is clear that $n_i(x)$ is well defined. Let χ_A be the characteristic function of A. We have

$$\int \chi_A(x)\frac{dp\circ T}{dp}\,dp(x) = \int \chi_A(x)\,dp\circ T(x) = \int \chi_A(T^{-1}x)\,dp(x)$$
$$= \int \chi_{T(A)}(x)\,dp(x) = p(T(A))$$
$$= p(A)\prod_{i=1}^{4}\beta_i^{n_i(x)} = \int \chi_A(x)\prod_{i=1}^{4}\beta_i^{n_i(x)}\,dp(x).$$

Since $\mathscr{B}$ is closed under finite intersections, a standard theorem in measure theory gives the formula for $(dp\circ T)/dp$.

To prove the rest of the lemma, we note that the β_i play only an auxiliary role in the definition of the n_i. Therefore the n_i are defined independent of the β_i, and they are intrinsic to the odometer. Uniqueness of the n_i is obvious.

We shall denote by P the group of finite permutations on $\mathbb{N}$. P acts on $\mathbf{X}$ in a canonical way, and $P \subseteq [T]$, the full group of T, and for $\tau \in P$, $(dp\circ\tau)/dp = 1$. This action is ergodic by the Hewitt-Savage 0-1 law.

Lemma 1.2. *Let $x \in \mathbf{X}$. If k is a positive integer such that $T^k(x)$ is a finite permutation of x, then*

$$\sum_{j=0}^{k-1} n_i(T^j(x)) = 0, \quad \textit{for } i = 1, 2, 3, 4.$$

Proof. Let $\tau \in P$ with $\tau(x) = T^k(x)$. For $j = 0$ to k, $T^j x$ differs from x in only a finite number of coordinates, say the first n coordinates. We let

$$A_0 = (x_1, x_2, \dots, x_n) \times \prod_{n+1}^{\infty}\mathbf{Z}_5, \quad A_1 = T^j(A_0), \qquad i = 0, 1, 2, \dots, k.$$

Then

$$\frac{p(A_{j+1})}{p(A_j)} = \prod_{i=1}^{4}\beta_i^{n_i(T^j(x))}, \qquad j = 0, 1, 2, \dots, k-1.$$

Multiplying these equations, we have

$$1 = \frac{p(A_k)}{p(A_0)} = \prod_{i=1}^{4}\beta_i^{\sum_j n_i(T^j x)}.$$

Since all β_i are multiplicatively independent, we have the result.

Lemma 1.3. *Let (F_t, Ω, ν) be a flow. There exists a Borel map $p(\omega, t)$ from $\Omega\times\mathbb{R}$ to $\mathbb{R}^+$ such that*

$$\frac{d\nu\circ F_t}{d\nu}(\omega) = p(\omega, t) \quad \textit{for all } t, \ \textit{a.e. } \omega.$$

Proof. See [10].

We shall use the same symbol $(d\nu\circ F_t)\omega/d\nu$ to denote $p(\omega, t)$. We shall also write

$$a(x) = n_1(x)\log\beta_1 + n_2(x)\log\beta_2, \quad b(x) = n_3(x)\log\beta_3 + n_4(x)\log\beta_4.$$

Theorem 1.4. *Let* (F_t, Ω, ν) *be a flow on a standard measure space. Let* S *be the transformation on* $(\mathbf{X} \times \Omega \times \mathbb{R}, p \times \nu \times e^{-s} ds)$ *defined by*

$$S(x, \omega, r) = \left(Tx, F_{a(x)}(\omega), r + \log \frac{d\nu \circ F_{a(x)}}{d\nu}(\omega) + b(x)\right).$$

Then the stable range of S *is the flow* (F_t, Ω, ν). *The transformation is ergodic if and only if the flow is ergodic.*

Proof. We denote the measure $p \times \nu \times e^{-s} ds$ by μ. In the construction of the stable range of S, $\widetilde{S}$ is defined on $(\mathbf{X} \times \Omega \times \mathbb{R} \times \mathbb{R}, \mu \times dt)$ by

$$\widetilde{S}(x, \omega, r, t) = \left(S(x, \omega, r), t + \log \frac{d\mu \circ S}{d\mu}(x, \omega, r)\right).$$

We calculate $(d\mu \circ S)/d\mu$. Let $A \in \mathscr{B}$, where $\mathscr{B}$ is as in Lemma 1.1. Then there exists $a, b \in \mathbb{R}$ such that $a(x) = a$ and $b(x) = b$ for all $x \in A$. Write

$$Z_{ab}(\omega, r) = \left(F_a(\omega), r + \log \frac{d\nu \circ F_a}{d\nu}(\omega) + b\right),$$

and $\mu' = \nu \times e^{-s} ds$. Z_{ab} is the composition of a measure preserving transformation and a translation which shrinks the measure by e^{-b}. So $\mu'(Z_{ab}(C)) = \mu'(C)e^{-b}$, for C Borel in $\Omega \times \mathbb{R}$. Now for such C,

$$\begin{aligned}\int_{A \times C} \frac{d\mu \circ S}{d\mu}(y)\, d\mu(y) &= \mu \circ S(A \times C) = \mu(T(A) \times Z_{ab}(C)) \\ &= p(T(A))\mu'(C)e^{-b} = e^a p(A)\mu'(C) \\ &= e^a \mu(A \times C) = \int_{A \times C} e^{a(x)}\, d\mu(y),\end{aligned}$$

where $x =$ the projection of y to A. Since such sets $A \times C$ are closed under finite intersections and generate the σ-algebra of $\mathbf{X} \times \Omega$, we see that, by Dinkin's theorem of measure theory, $(d\mu \circ S)(y)/d\mu = e^{a(x)}$. Hence $\widetilde{S}(x, \omega, r, t) = (S(x, \omega, r), t + a(x))$. Let $\tau \in P$, and let $k(x)$ be the integral valued function such that $\tau(x) = T^{k(x)}x$, for all $x \in \mathbf{X}$. Then if x is such that $k(x) \geq 0$, and abbreviating $\sum_{i=0}^{j} a(T^i x)$ to $s(j)$, we have

$$\begin{aligned}\widetilde{S}^{k(x)}(x, \omega, r, t) &= S^{k(x)}(x, \omega, r), t + s(k(x) - 1) \\ &= \Bigg(T^{k(x)}x, F_{s(k(x)-1)}(\omega), \\ &\qquad r + \log\left(\prod_{j=0}^{k(x)-1} \frac{dF_{a(T^j x)} \circ \mu}{d\mu}(F_{s(j-1)}(\omega))\right) + \sum_{j=0}^{k(x)-1} b(T^j x), t\Bigg) \\ &= (\tau x, \omega, r, t) \quad \text{a.e. } (x, \omega, r, t),\end{aligned}$$

by Lemma 1.2. By considering τ^{-1} applied on $\tau(x)$, we see that this is also valid for x with $k(x) \leq 0$.

Let A be an $\widetilde{S}$ invariant set in $\mathbf{X} \times \Omega \times \mathbb{R} \times \mathbb{R}$. Then A is $p \times 1 \times 1 \times 1 \times 1$ invariant. As P is ergodic on $\mathbf{X}$, $A = \mathbf{X} \times B$ for some measurable $B \subseteq \Omega \times \mathbb{R} \times \mathbb{R}$. B is invariant under

$$(\omega, r, t) \to \left(F_{a(x)}(\omega), r + \log \frac{d\mu \circ F_{a(x)}}{d\mu}(\omega) + b(x), t + a(x)\right)$$

for all $x \in \mathbf{X}$. Note that each of the following sets has positive measure:

$$\{x_1\colon a(x_1) = -\log \beta_1 \text{ and } b(x_1) = 0\},$$
$$\{x_2\colon a(x_2) = -\log \beta_2 \text{ and } b(x_2) = 0\},$$
$$\{x_3\colon a(x_3) = 0 \text{ and } b(x_3) = -\log \beta_3\},$$
$$\{x_4\colon a(x_4) = 0 \text{ and } b(x_4) = -\log \beta_4\}.$$

This implies B is invariant under

$$(\omega, r, t) \to \left(F_a(\omega), r + \log \frac{d\mu \circ F_a}{d\mu}(\omega) + b, t + a\right),$$

where $a \in G_1$, $b \in G_2$, and G_1, G_2 are the groups generated by $\log \beta_1$, $\log \beta_2$, and $\log \beta_3$, $\log \beta_4$ respectively. Since these groups are dense in $\mathbb{R}$, we have $B = \{(\omega, r, t)\colon r \in \mathbb{R}, (\omega, t) \in C\}$ for some Borel set C of $\Omega \times \mathbb{R}$, and C is invariant under

$$\alpha_a\colon (\omega, t) \to (F_a(\omega), t + a) \quad \text{for all } a \in G_1.$$

Since the action of $\mathbb{R}$ on $L^\infty(\Omega)$ induced by F_t is σ-strongly continuous, i.e., $t_n \to t$ implies $f \circ F_{t_n} \to f \circ F_t$ σ-strongly for $f \in L^\infty(\Omega)$, the action of $\mathbb{R}$ on $L^\infty(\Omega \times \mathbb{R})$ induced by α_a, $a \in \mathbb{R}$, is continuous. Thus C is invariant under α_a for all $a \in \mathbb{R}$. Such invariant sets are in one-to-one correspondence with the Borel subsets of Ω, via the map $(\omega, t) \to F_{-t}(\omega)$. Hence

$$L^\infty(\mathbf{X} \times \Omega \times \mathbb{R} \times \mathbb{R})^{\widetilde{S}} \cong L^\infty(\Omega),$$

via the isomorphism $\sigma\colon f \to g \in L^\infty(\Omega)$, where g is the function which satisfies the equation

$$g(F_{-t}(\omega)) = f(x, \omega, r, t) \quad \text{a.e. } (x, \omega, r, t).$$

The point realization of the canonical $\mathbb{R}$ action is then F_s, as required.

The proof of the ergodicity part is standard and is omitted.

Remark. In case the flow we start with is measure preserving, the construction can be achieved with a three point odometer. We describe the construction in the next theorem.

Theorem 1.5. *Suppose that (F_t, Ω, ν) is measure preserving, i.e., $\nu \circ F_t = \nu$ for all $t \in \mathbb{R}$. Let $\mathbf{X}_3 = \prod_1^\infty \mathbf{Z}_3$ and $p = \prod_1^\infty (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$. Then the transformation S on $\mathbf{X}_3 \times \Omega$ defined by $S(x, \omega) = (Tx, F_{a(x)}\omega)$, where $a(x) = \log(dp \circ T)(x)/dp$, has stable range equal to (F_t, Ω, ν).*

Proof. The proof is analogous to but simpler than the proof of the main theorem. The reader can also refer to [4] for a proof.

In the following theorem, the stable range functor is denoted by Φ.

Theorem 1.6. *The construction of the transformation from the flow in Theorem 1.4 is a functor Ψ from the category of ergodic flows with conjugations to the category of ergodic transformations with orbit transporting isomorphisms. Ψ is a right inverse functor to the stable range functor, i.e., $\Phi \circ \Psi$ is naturally isomorphic to the identity functor.*

Proof. Suppose we have two flows $F = (F_t, \Omega, \nu)$, $G = (G_t, \Omega_1, \nu_1)$, and $\phi\colon \Omega \to \Omega_1$ a conjugation between them. Let (S, Y, μ), (S_1, Y_1, μ_1) be respectively $\Psi(F)$, $\Psi(G)$. Define $\Psi(\Phi)\colon Y = (\mathbf{X} \times \Omega \times \mathbb{R}, \mu) \to Y_1 = (\mathbf{X} \times \Omega_1 \times \mathbb{R}, \mu_1)$ by

$$\Psi(\Phi)(x, \omega, r) = (x, \phi(\omega), r + p(\phi(\omega))),$$

where

$$p(\omega') = \log \frac{d\nu_1}{d\phi^*\nu}(\omega'), \qquad \omega' \in \Omega_1.$$

Then $\Psi(\phi)$ is an isomorphism, and

$$\begin{aligned}\Psi(\phi) \circ S(x, \omega, r) &= \Psi(\phi)\left(Tx, F_{a(x)}(\omega), r + b(x) + \log \frac{d\nu \circ F_{a(x)}}{d\nu}(\omega)\right)\\ &= \left(Tx, \phi \circ F_{a(x)}(\omega), r + b(x) + \log \frac{d\nu \circ F_{a(x)}}{d\nu}(\omega)\right) + p(\phi(F_{a(x)}\omega))\\ &= \left(Tx, G_{a(x)}(\phi(\omega)), r + b(x) + \log \frac{d\nu_1 \circ G_{a(x)}}{d\nu_1}(\phi(\omega)) + p(\phi(\omega))\right)\\ &= S_1 \circ \Psi(\phi)(x, \omega, r) \quad \text{a.e. } (x, \omega, r) \in \mathbf{X} \times \Omega \times \mathbb{R}.\end{aligned}$$

Hence $\Psi(\phi)$ not only preserves the orbit of S, but the conjugates between S and S_1 as well.

Given two composable conjugations ϕ, ψ between flows, it is easy to see that $\Psi(\phi_1 \circ \phi_2) = \Psi(\phi_1)\Psi(\phi_2)$. Hence Ψ is a functor.

To prove the last statement, let (F_t, Ω, ν) be in $\mathscr{F}$. Notations as above, we have $\Psi(F) = (S, Y, \nu)$. Let $A = L^\infty(Y \times \mathbb{R}, \mu \times ds)^{\widetilde{S}}$. Then there exist isomorphisms

$$\sigma_A\colon A \to L^\infty(I, m) \quad \text{and} \quad \sigma\colon A \to L^\infty(\Omega, \nu),$$

where σ_A is as defined in the construction of the stable range functor and σ is defined in the last part of Theorem 1.4. Since F and $\Phi\Psi(F)$ are the point realizations of the flow on A via σ and σ_A respectively, the point realization of $\sigma_A\sigma^{-1}$ via the identity map is a conjugation between the flows F and $\Phi\Psi(F)$. Denote it by n_F. We prove that it is a natural isomorphism between the identity functor and $\Phi\Psi$.

Let $\phi\colon (F_t, \Omega, \nu) \to (G_t, \Omega_1, \nu_1)$ be a conjugation. Let

$$B = L^\infty(Y_1 \times \mathbb{R}, \mu_1 \times ds)^{\widetilde{S_1}}.$$

Define $\widetilde{\Psi(\phi)}$ as in the construction of Φ. $\widetilde{\Psi(\phi)}$ induces a map $\alpha\colon A \to B$ by $\alpha(f) = f \circ \widetilde{\Psi(\phi)}^{-1}$. Let $\sigma_1\colon B \to L^\infty(\Omega_1, \nu_1)$ be the canonical identification map similar to σ. Then

$$\begin{aligned}(\sigma_1\alpha\sigma^{-1})(\sigma\sigma_A^{-1}) = \sigma_1\alpha\sigma_A^{-1} &= \sigma_1\sigma_B^{-1}(\sigma_B\alpha\sigma_A^{-1})\sigma_A\sigma_A^{-1}\\ &= (\sigma_1\sigma_B^{-1})(\sigma_B\alpha\sigma_A^{-1}).\end{aligned}$$

Hence the following diagram commutes

$$\begin{array}{ccc} L^\infty(I, m) & \xrightarrow{\sigma_B\alpha\sigma_A^{-1}} & L^\infty(I, m) \\ \sigma\sigma_A^{-1} \downarrow & & \downarrow \sigma_1\sigma_B^{-1} \\ L^\infty(\Omega, \nu) & \xrightarrow[\sigma_1\alpha\sigma^{-1}]{} & L^\infty(\Omega_1, \nu_1). \end{array}$$

The point realization of $\sigma_B \alpha \sigma_A^{-1}$ via the identity map is $\Phi\Psi(\phi)$, while it is easy to see that the point realization of $\sigma_1 \alpha \sigma^{-1}$ via the identity map is ϕ itself. By taking point realization of the diagram, we get $n_G \circ \phi = \Phi\Psi(\phi) \circ n_F$.

Remark. As the canonical mod homomorphism from the normalizer group of a transformation to the centralizer of its associated flow is contained in the stable range functor, the above theorem implies that it is surjective and moreover is split.

It is not known, but likely, that $\Psi\Phi$ is isomorphic to the identity functor on $\mathscr{X}$. The problem is of course with the morphisms, since Krieger's theorem proves that the objects $\Psi\Phi(T)$ and T are isomorphic.

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School of Mathematics, University of New South Wales, N.S.W., Australia